

Polynomial inequalities for non-commuting operators

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April 14, 2010

Abstract

We prove an inequality for polynomials applied in a symmetric way to non-commuting operators.

1 Introduction

J. von Neumann [9] proved an inequality about the norm of a polynomial applied to a contraction on a Hilbert space H . Let \mathbb{D} be the unit disk and \mathbb{T} the unit circle in \mathbb{C} , and for any polynomial p let $\|p\|_X$ be the supremum of the modulus of p on the set X . The result is that

$$T \in \mathcal{B}(H), \|T\| \leq 1 \Rightarrow \|p(T)\| \leq \|p\|_{\mathbb{D}}. \quad (1.1)$$

For polynomials $p(z) = p(z_1, z_2, \dots, z_n) = \sum_{|\alpha| \leq N} c_\alpha z^\alpha$ in n variables we use the standard multi-index notation (where $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ has $0 \leq \alpha_j \in \mathbb{Z}$ for $1 \leq j \leq n$, $|\alpha| = \sum_{j=1}^n \alpha_j$, $z^\alpha = \prod_{j=1}^n z_j^{\alpha_j}$). There is an obvious way of applying p to an n -tuple $T = (T_1, T_2, \dots, T_n)$ of *commuting* operators $T_j \in \mathcal{B}(H)$ ($1 \leq j \leq n$), namely

$$p(T) = p(T_1, T_2, \dots, T_n) = \sum_{|\alpha| \leq N} c_\alpha T^\alpha$$

*Partially supported by National Science Foundation Grants DMS 0501079 and DMS 0966845

(with $T^\alpha = \prod_{j=1}^n T_j^{\alpha_j}$ and $T_j^0 = I$).

T. Andô [2] proved an extension of von Neumann's inequality to pairs of commuting contractions.

Theorem 1.1 (Andô). *If $T_1, T_2 \in \mathcal{B}(H)$, $\max(\|T_1\|, \|T_2\|) \leq 1$, $T_1 T_2 = T_2 T_1$ and $p(z) = p(z_1, z_2)$ is a polynomial, then*

$$\|p(T_1, T_2)\| \leq \|p\|_{\mathbb{D}^2}.$$

The purpose of this note is to look for analogues of Andô's inequality that are satisfied by *non-commuting* operators. For a polynomial p in n variables and an n -tuple of operators $T = (T_1, \dots, T_n)$ we define $p_{\text{sym}}(T)$ to be a symmetrized version of p applied to T (we make this precise in Section 2). We are looking for results of the form:

For all n -tuples T of operators in a certain set, there is a set K_1 in \mathbb{C}^n such that

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{K_1}. \quad (1.2)$$

and

For all n -tuples T of operators in a certain set, there is a set K_2 in \mathbb{C}^n and a constant M such that

$$\|p_{\text{sym}}(T)\| \leq M \|p\|_{K_2}. \quad (1.3)$$

Our main result is:

Theorem 4.6 *There are positive constants M_n and R_n such that, whenever $T = (T_1, T_2, \dots, T_n) \in \mathcal{B}(H)^n$ satisfies*

$$\left\| \sum_{i=1}^n \zeta_i T_i \right\| \leq 1 \quad \forall \zeta_i \in \overline{\mathbb{D}},$$

and p is a polynomial in n variables, then

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{R_n \overline{\mathbb{D}}^n} \quad (1.4)$$

$$\|p_{\text{sym}}(T)\| \leq M_n \|p\|_{\overline{\mathbb{D}}^n}. \quad (1.5)$$

Moreover, one can choose $R_2 = 1.85$, $R_3 = 2.6$, $M_2 = 4.1$ and $M_3 = 16.6$.

2 Tuples of noncommuting contractions

There are several natural ways one might apply a polynomial $p(z_1, z_2)$ in two variables to pairs $T = (T_1, T_2) \in \mathcal{B}(H)^2$ of operators. A simple case is for polynomials of the form $p(z_1, z_2) = p_1(z_1) + p_2(z_2)$ where we could naturally consider $p(T_1, T_2)$ to mean $p_1(T_1) + p_2(T_2)$.

A recent result of Drury [5] is that if $p(z_1, z_2) = p_1(z_1) + p_2(z_2)$, $T_1, T_2 \in \mathcal{B}(H)$ (no longer necessarily commuting), $\max(\|T_1\|, \|T_2\|) \leq 1$, then

$$\|p(T_1, T_2)\| \leq \sqrt{2}\|p\|_{\mathbb{D}^2}. \quad (2.1)$$

Moreover, Drury [5] shows that the constant $\sqrt{2}$ is best possible.

One way to apply a polynomial $p(z_1, z_2) = \sum_{j,k=0}^n a_j z_1^j z_2^k$ to two non-commuting operators T_1 and T_2 is by mapping each monomial $z_1^j z_2^k$ to the average over all possible products of j number of T_1 and k number of T_2 , and then extend this map by linearity to all polynomials. We use the notation $p_{\text{sym}}(T_1, T_2)$ and the formula

$$p_{\text{sym}}(T_1, T_2) = \sum_{j,k=0}^n \frac{a_j}{\binom{j+k}{j}} \sum_{S \in \mathcal{P}(j+k, j)} \prod_{i=1}^{j+k} T_{2-\chi_S(i)}$$

where $\mathcal{P}(j+k, j)$ denotes the subsets of $\{1, 2, \dots, j+k\}$ of cardinality j . The empty product, which arises for $j = k = 0$, should be taken as the identity operator. The notation $\prod_{i=1}^{j+k} T_{2-\chi_S(i)}$ is intended to mean the ordered product

$$T_{2-\chi_S(1)} T_{2-\chi_S(2)} \cdots T_{2-\chi_S(j+k)},$$

and $\chi_S(i)$ denotes the indicator function of S .

Remarks 2.1. The operation $p \mapsto p_{\text{sym}}(T_1, T_2)$ is not an algebra homomorphism (from polynomials to operators). It is a linear operation and does not respect squares in general.

For example, if $p(z_1, z_2) = z_1^2 + z_2^2$, then

$$p_{\text{sym}}(T_1, T_2) = T_1^2 + T_2^2$$

but for $q(z_1, z_2) = (p(z_1, z_2))^2 = z_1^4 + z_2^4 + 2z_1^2 z_2^2$ we have

$$(p_{\text{sym}}(T_1, T_2))^2 = T_1^4 + T_2^4 + T_1^2 T_2^2 + T_2^2 T_1^2 \neq q_{\text{sym}}(T_1, T_2)$$

in general.

Similarly for $p(z_1, z_2) = 2z_1 z_2$ and

$$q(z_1, z_2) = (p(z_1, z_2))^2 = 4z_1^2 z_2^2,$$

$$p_{\text{sym}}(T_1, T_2) = T_1 T_2 + T_2 T_1,$$

$$(p_{\text{sym}}(T_1, T_2))^2 = T_1 T_2 T_1 T_2 + T_1 T_2^2 T_1 + T_2 T_1^2 T_2 + T_2 T_1 T_2 T_1 \neq q_{\text{sym}}(T_1, T_2)$$

in general.

However in the very restricted situation that $p(z_1, z_2) = \alpha + \beta z_1 + \gamma z_2$ and $q = p^m$, then we do have $q_{\text{sym}}(T_1, T_2) = (p_{\text{sym}}(T_1, T_2))^m$.

The symmetrizing idea generalizes in the obvious way to $n > 2$ variables. We will use the notation $p_{\text{sym}}(T)$ for n -tuples $T \in \mathcal{B}(H)^n$ for $n \geq 2$.

3 Example

The analogue of Andô's inequality for $n \geq 3$ commuting Hilbert space contractions and polynomials norms on \mathbb{D}^n is known to fail (see Varopoulos [10], Crabb & Davie [4], Lotto & Steger [7], Holbrook [6]).

The explicit counterexamples of Kaijser & Varopoulos [10], and Crabb & Davie [4] have $p(T)$ nilpotent (and so of spectral radius 0). While the examples of Lotto & Steger [7] and Holbrook [6] do not have this property, they are obtained by perturbing examples where $p(T)$ is nilpotent (and so $p(T)$ has relatively small spectral radius).

It is not known whether there is a constant C_n so that the multi-variable inequality

$$\|p(T)\| = \|p(T_1, T_2, \dots, T_n)\| \leq C_n \|p\|_{\mathbb{D}^n} \quad (3.1)$$

holds for all polynomials $p(z)$ in n variables and for all n -tuples T of commuting Hilbert space contractions. However, it is well-known that a spectral radius version of Andô's inequality is true — indeed, it holds in any Banach algebra.

Proposition 3.1. *If p is a polynomial in n variables and $T = (T_1, T_2, \dots, T_n)$ is an n -tuple of commuting elements in a Banach algebra, each with norm at most one, then*

$$\rho(p(T)) = \lim_{m \rightarrow \infty} \|(p(T))^m\|^{1/m} \leq \|p\|_{\mathbb{D}^n} \quad (3.2)$$

Proof. We consider a fixed n . It follows from the Cauchy integral formula, that if $\max_{1 \leq j \leq n} \|T_j\| \leq r < 1$, then

$$\|p(T)\| = \|p(T_1, T_2, \dots, T_n)\| \leq C_r \|p\|_{\mathbb{D}^n} \quad (3.3)$$

for a constant C_r depending on r (and n).

To see this write

$$p(T) = \frac{1}{(2\pi i)^n} \int_{\zeta \in \mathbb{T}^n} \prod_{j=1}^n p(\zeta_j) \prod_{j=1}^n (\zeta_j - T_j)^{-1} d\zeta_1 d\zeta_2 \dots d\zeta_n$$

and estimate with the triangle inequality. This shows that $C_r = (1 - r)^{-n}$ will work.

Applying (3.3) to powers of p and using the spectral radius formula, we get

$$\rho(p(T)) \leq \|p\|_{\mathbb{D}^n},$$

(provided $\max_{1 \leq j \leq n} \|T_j\| \leq r < 1$). However, for the general case $\max_{1 \leq j \leq n} \|T_j\| = 1$, we can apply this to rT to get

$$\rho(p(T)) = \lim_{r \rightarrow 1^-} \rho(p(rT)) \leq \|p\|_{\infty}. \quad \square$$

Example 3.2. Let $p(z, w) = (z - w)^2 + 2(z + w) + 1 = z^2 + w^2 - 2zw + 2(z + w) + 1$,

$$T_1 = \begin{pmatrix} \cos(\pi/3) & \sin(\pi/3) \\ \sin(\pi/3) & -\cos(\pi/3) \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix},$$

$$T_2 = \begin{pmatrix} \cos(\pi/3) & -\sin(\pi/3) \\ -\sin(\pi/3) & -\cos(\pi/3) \end{pmatrix}.$$

Note that $\|p\|_{\mathbb{D}^2} \geq p(1, -1) = 5$. To show that $\|p\|_{\mathbb{D}^2} \leq 5$, consider the homogeneous polynomial

$$q(z_1, z_2, z_3) = z_1^2 + z_2^2 + z_3^2 - 2z_1z_2 - 2z_1z_3 - 2z_2z_3$$

and observe first that $p(z, w) = q(z, w, -1)$. Moreover

$$\|p\|_{\mathbb{D}^2} = \|p\|_{\mathbb{T}^2} = \|q\|_{\mathbb{T}^3} = \|q\|_{\mathbb{D}^3},$$

by homogeneity of q and the maximum principle. Holbrook [6, Proposition 2] gives a proof that $\|q\|_{\mathbb{D}^3} = 5$.

We have

$$\begin{aligned} p_{\text{sym}}(T_1, T_2) &= (T_1 - T_2)^2 + 2(T_1 + T_2) + I \\ &= \begin{pmatrix} 0 & \sqrt{3} \\ \sqrt{3} & 0 \end{pmatrix}^2 + 2 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + I \\ &= \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} \end{aligned}$$

So $\|p_{\text{sym}}(T_1, T_2)\| = 6 > 5 = \|p\|_{\mathbb{D}^2}$.

Remark 3.3. The example has hermitian T_1 and T_2 and a polynomial with real coefficients and yet $\rho(p_{\text{sym}}(T_1, T_2)) > \|p\|_{\mathbb{D}^2}$. Thus even Proposition 3.1 does not hold for non-commuting pairs.

One can show that for the polynomial p of Example 3.2, one has the inequality

$$\|p_{\text{sym}}(T_1, T_2)\| \leq (1 + 4\sqrt{2}) \approx 6.66$$

for all contractions T_1 and T_2 . This estimate is at least an improvement over using the sum of the absolute values of the coefficients of p , so one is led to ask how well can one bound $\|p_{\text{sym}}(T)\|$ for general p ?

$$4 \quad \left\| \sum \zeta_i T_i \right\| \leq 1$$

In this section, we shall consider n -tuples $T = (T_1, \dots, T_n)$ of operators, not assumed to be commuting, and we shall make the standing assumption:

$$\left\| \sum_{i=1}^n \zeta_i T_i \right\| \leq 1 \quad \forall \zeta_i \in \overline{\mathbb{D}}. \quad (4.1)$$

This will hold, for example, if the condition

$$\sum_{i=1}^n \|T_i\| \leq 1 \quad (4.2)$$

holds. We wish to derive bounds on $\|p_{\text{sym}}(T)\|$. We start with the following lemma:

Lemma 4.1. *If $S \in \mathcal{B}(H)$ and $\|S\| < 1$ then*

$$\Re((I + S)(I - S)^{-1}) \geq 0.$$

Proof.

$$\begin{aligned} & 2\Re((I + S)(I - S)^{-1}) \\ &= (I - S^*)^{-1}(I + S^*) + (I + S)(I - S)^{-1} \\ &= (I - S^*)^{-1}[(I + S^*)(I - S) + (I - S^*)(I + S)](I - S)^{-1} \\ &= 2(I - S^*)^{-1}[I - S^*S](I - S)^{-1} \\ &\geq 0. \end{aligned}$$

□

If $p(z) = \sum c_\alpha z^\alpha$, define

$$\Gamma p(z) = \sum c_\alpha \frac{\alpha!}{|\alpha|!} z^\alpha \quad (4.3)$$

(as usual, $\alpha!$ means $\alpha_1! \cdots \alpha_n!$). We let Λ denote the inverse of Γ :

$$\Lambda \sum d_\alpha z^\alpha = \sum d_\alpha \frac{|\alpha|!}{\alpha!} z^\alpha.$$

Proposition 4.2. *Let $T = (T_1, T_2, \dots, T_n) \in \mathcal{B}(H)^n$ satisfy (4.1) and $p(z)$ be a polynomial in n variables. Then*

$$\|p_{\text{sym}}(T)\| \leq \|\Gamma p\|_{\mathbb{D}^n}. \quad (4.4)$$

Proof. We first restrict to the case

$$\zeta = (\zeta_1, \zeta_2, \dots, \zeta_n) \in \mathbb{T}^n \Rightarrow \|\zeta \cdot T\| = \left\| \sum_{j=1}^n \zeta_j T_j \right\| < 1$$

and hence by Lemma 4.1 the operator

$$(I + \zeta \cdot T)(I - \zeta \cdot T)^{-1} = (I + \zeta \cdot T) \sum_{j=0}^{\infty} (\zeta \cdot T)^j = I + 2 \sum_{j=1}^{\infty} (\zeta \cdot T)^j$$

has positive real part

$$\begin{aligned} K(\zeta, T) &= \Re((I + \zeta \cdot T)(I - \zeta \cdot T)^{-1}) \\ &= I + \sum_{j=1}^{\infty} (\zeta \cdot T)^j + \sum_{j=1}^{\infty} (\bar{\zeta} \cdot T^*)^j \\ &= 2\Re \left[\sum_{\alpha_1, \dots, \alpha_n=0}^{\infty} \frac{|\alpha|!}{\alpha!} \zeta^\alpha (z^\alpha)_{\text{sym}}(T) \right] - I. \end{aligned}$$

We can compute that for polynomials $p(z) = p(z_1, z_2, \dots, z_n)$,

$$p_{\text{sym}}(T) = \int_{\mathbb{T}^n} \Gamma p(\zeta) K(\bar{\zeta}, T) d\sigma(\zeta)$$

with $d\sigma$ indicating normalised Haar measure on the torus \mathbb{T}^n (and $\bar{\zeta} = (\bar{\zeta}_1, \bar{\zeta}_2, \dots, \bar{\zeta}_n)$).

As

$$K(\bar{\zeta}, T) d\sigma(\zeta)$$

is a positive operator valued measure on \mathbb{T}^n , we then have a positive unital linear map $C(\mathbb{T}^n) \rightarrow \mathcal{B}(H)$ given by $f \mapsto \int_{\mathbb{T}^n} f(\zeta) K(\bar{\zeta}, T) d\sigma(\zeta)$. As this map is then of norm 1, we can conclude

$$\|p_{\text{sym}}(T)\| \leq \|\Gamma p\|_{\mathbb{D}^n}.$$

For the remaining case $\sup_{\zeta \in \mathbb{T}^n} \|\zeta \cdot T\| = 1$, we have

$$\|p_{\text{sym}}(T)\| = \lim_{r \rightarrow 1^-} \|p_{\text{sym}}(rT)\| \leq \|\Gamma p\|_{\mathbb{D}^n}. \quad \square$$

Remark 4.3. The technique of the above proof is derived from methods of [8].

Now we want to estimate $\|\Gamma p\|_{\mathbb{D}^n}$.

Proposition 4.4. *For each $n \geq 2$ there is a constant M_n so that*

$$\|\Gamma p\|_{\mathbb{D}^n} \leq M_n \|p\|_{\mathbb{D}^n}.$$

Moreover,

$$\begin{aligned} M_2 &\leq 4.07 \\ M_3 &\leq 16.6 \end{aligned}$$

Proof. Define

$$J(\eta) = \sum_{\alpha_1=0, \dots, \alpha_n=0}^{\infty} \frac{\alpha!}{|\alpha|!} \eta^\alpha. \quad (4.5)$$

Then

$$\Gamma p(z) = \int_{\mathbb{T}^n} p(\zeta) [J(z_1 \bar{\zeta}_1, \dots, z_n \bar{\zeta}_n)] d\sigma(\zeta). \quad (4.6)$$

To use (4.6), we break J into two parts — the sum J_0 where the minimum of the α_i is 0, and the remaining terms J_1 .

$$J_1(\eta) = \sum_{\alpha_1=1, \dots, \alpha_n=1}^{\infty} \frac{\alpha!}{|\alpha|!} \eta^\alpha.$$

Case: $n = 2$. Here,

$$\int_{\mathbb{T}^2} p(\zeta) J_0(z_1 \bar{\zeta}_1, z_2 \bar{\zeta}_2) d\sigma(\zeta) = p(z_1, 0) + p(0, z_2) - p(0, 0). \quad (4.7)$$

So the norm of the left-hand side of (4.7) is dominated by $3\|p\|_{\mathbb{D}^2}$.

For J_1 , we will use the estimate

$$\left| \int_{\mathbb{T}^2} p(\zeta) J_1(z_1 \bar{\zeta}_1, z_2 \bar{\zeta}_2) d\sigma(\zeta) \right| \leq \|p\|_\infty \|J_1\|_{L^1} \leq \|p\|_\infty \|J_1\|_{L^2}.$$

We have

$$\begin{aligned} \|J_1\|_{L^2}^2 &= \sum_{\alpha_1, \alpha_2=1}^{\infty} \left(\frac{\alpha_1! \alpha_2!}{(\alpha_1 + \alpha_2)!} \right)^2 \\ &= \sum_{\alpha_1=1}^{\infty} \frac{1}{(\alpha_1 + 1)^2} + \sum_{\alpha_2=2}^{\infty} \frac{1}{(\alpha_2 + 1)^2} + \sum_{\alpha_1, \alpha_2=2}^{\infty} \left(\frac{\alpha_1! \alpha_2!}{(\alpha_1 + \alpha_2)!} \right)^2 \\ &\leq \left(\frac{\pi^2}{3} - \frac{9}{4} \right) + \sum_{k=4}^{\infty} (k-3) \left(\frac{2}{k(k-1)} \right)^2 \\ &\leq (1.069)^2. \end{aligned}$$

(In the penultimate line, we let $k = \alpha_1 + \alpha_2$; there are $k-3$ terms with this sum, and the largest they can be is when either α_1 or α_2 is 2.) Adding the two estimates, we get $M_2 \leq 4.07$.

Case: $n = 3$. Again, we estimate the contributions of J_0 and J_1 separately. We have

$$\begin{aligned} &\int p(\zeta) J_0(z_1 \bar{\zeta}_1, z_2 \bar{\zeta}_2, z_3 \bar{\zeta}_3) d\sigma(\zeta) \\ &= \Gamma p(0, z_2, z_3) + [\Gamma p(z_1, 0, z_3) - p(0, 0, z_3)] \\ &\quad + [\Gamma p(z_1, z_2, 0) - p(z_1, 0, 0) - p(0, z_2, 0) + p(0, 0, 0)] \end{aligned}$$

where we have had to subtract some terms to avoid double-counting. Thus the contribution of J_0 is at most $3M_2 + 4$.

To calculate the contribution of J_1 , we make the following estimate on $\|J_1\|_{L^2}$, which is valid for all $n \geq 3$:

We want to bound

$$\sum_{\alpha_1=1, \dots, \alpha_n=1}^{\infty} \left(\frac{\alpha!}{|\alpha|!} \right)^2 \tag{4.8}$$

Let $k = |\alpha|$ in (4.8). Note first that the number of terms for each k is the number of ways of writing k as a sum of n distinct positive integers (order matters), and this is exactly $\binom{k-1}{n-1}$. Moreover, as each α_i is at least 1, we have

$$\frac{\alpha!}{|\alpha|!} \leq \frac{1}{k(k-1) \cdots (k-n+2)}.$$

Therefore (4.8) is bounded by

$$\begin{aligned} & \sum_{k=n}^{\infty} \binom{k-1}{n-1} \left(\frac{1}{k(k-1)\cdots(k-n+2)} \right)^2 \\ &= \sum_{k=n}^{\infty} \frac{k-n+1}{(n-1)!k} \frac{1}{k(k-1)\cdots(k-n+2)}. \end{aligned}$$

The terms on the right-hand side of (4.9) decay like $1/k^{n-1}$, so the series converges for all $n \geq 3$. When $n = 3$, the series is

$$\sum_{k=3}^{\infty} \frac{k-2}{2k^2(k-1)} \leq (0.381)^2.$$

Therefore $M_3 \leq 3M_2 + 4.381 < 16.59$.

We now proceed by induction on n . The contribution from J_0 is dominated by applying Γ to the restriction of p to the slices with one or more coordinates equal to 0, and these are bounded by the inductive hypothesis. The contribution from J_1 is bounded by (4.8). □

We have proved that the polydisk is an M -spectral set for T ; we can make the constant one by enlarging the domain.

Proposition 4.5. *There is a constant R_n so that*

$$\|\Gamma p\|_{\mathbb{D}^n} \leq \|p\|_{R_n \mathbb{D}^n}. \quad (4.9)$$

Moreover,

$$\begin{aligned} R_2 &\leq 1.85 \\ R_3 &\leq 2.6 \end{aligned}$$

Proof. Let $L(\eta) = 2\Re J(\eta) - 1$. Adding terms that are not conjugate analytic powers of ζ inside the bracket in (4.6) will not change the value of the integral, so, writing $z\bar{\zeta}$ for the n -tuple $(z_1\bar{\zeta}_1, \dots, z_n\bar{\zeta}_n)$, we get

$$\Gamma p(z) = \int_{\mathbb{T}^n} p(\zeta) [L(z\bar{\zeta})] d\sigma(\zeta). \quad (4.10)$$

As L is real and has integral 1, if we can choose r_n so that if $|z_i| \leq r_n$ for each i then $L(z\bar{\zeta})$ is non-negative for all ζ , then its L^1 norm would equal its integral, and so we would get from (4.10) that

$$|\Gamma p(z)| \leq \|p\|_{\mathbb{D}^n}.$$

Letting $R_n = 1/r_n$ gives (4.9). As the series (4.5) converges absolutely for all $\eta \in \mathbb{D}^n$, and $L(0) = 1$, the existence of some r_n now follows by continuity.

Let us turn now to obtaining quantitative estimates.

Case: $n = 2$. Adding terms to J that are not analytic will not affect the integral (4.10), so let us consider

$$L'(\eta) = \Re \left[\frac{1 + \eta_1}{1 - \eta_1} \right] \cdot \Re \left[\frac{1 + \eta_2}{1 - \eta_2} \right] - \sum_{\alpha_1=1, \alpha_2=1}^{\infty} \left(1 - \frac{\alpha!}{|\alpha|!}\right) (\eta_1^{\alpha_1} - \bar{\eta}_1^{\alpha_1})(\eta_2^{\alpha_2} - \bar{\eta}_2^{\alpha_2}).$$

Then L' has integral 1 and (4.10) is unchanged if L is replaced by L' . So we wish to find the largest r so that L' is positive on $r\mathbb{D}^2$.

It can be checked numerically that $r = 0.5406$ works, so the best R_2 is smaller than the reciprocal of 0.5406, which is less than 1.85.

Case: $n = 3$. (By hand).

Our strategy will be to simply estimate each non-constant term in L by a function of r , add them up, and see how small r must be for all these terms to be less than 1.

First, let us estimate the terms from $J_0 - 1$, i.e. those terms with either one or two of the α_i 's equal to 0. We can write

$$J_0(\eta) - 1 = [\eta_1 + \sum_{\alpha_2=1, \alpha_3=1}^{\infty} \frac{\alpha_2! \alpha_3!}{(\alpha_2 + \alpha_3)!} \eta_2^{\alpha_2} \eta_3^{\alpha_3}] + \dots,$$

where the \dots mean two more terms with the indices $(1, 2, 3)$ permuted. Therefore if each $|\eta_i| \leq r$, we have

$$\begin{aligned} 2\Re[J_0(\eta) - 1] &\leq 6r + 6 \sum_{\alpha_2=1, \alpha_3=1}^{\infty} \frac{\alpha_2! \alpha_3!}{(\alpha_2 + \alpha_3)!} r^{\alpha_2 + \alpha_3} \\ &\leq 6r + 6(r^2/2 + 2r^3/3) + 6 \sum_{k=4}^{\infty} r^k (k-1) \frac{1}{k(k-1)} \\ &= 6r + 6(r^2/2 + 2r^3/3) + 6[-\log(1-r) - r - r^2/2 - r^3/3]. \end{aligned}$$

The contribution to L from J_1 , where all the indices are at least 1, is at most

$$\begin{aligned} 2\Re[J_1(\eta)] &\leq 2 \sum_{\alpha_1=1, \alpha_2=1, \alpha_3=1}^{\infty} \frac{\alpha!}{|\alpha|!} r^{|\alpha|} \\ &\leq 2 \sum_{k=3}^{\infty} r^k \binom{k-1}{2} \frac{1}{k(k-1)} \\ &= \frac{r^3}{1-r} + 2[\log(1-r) + r + r^2/2]. \end{aligned}$$

Adding the two terms together, we get

$$L(\eta) \geq 1 - [2r + r^2 + 2r^3 + \frac{r^3}{1-r} - \log(1-r)],$$

and this is positive if $r \leq .152$. So letting R_3 be the reciprocal of this root, which is less than 6.6, will work.

Case: $n = 3$. (Computer-aided)

As in the case $n = 2$, we consider the kernel

$$\begin{aligned} L'(\eta) &= \Re \left[\frac{1 + \eta_1}{1 - \eta_1} \right] \cdot \Re \left[\frac{1 + \eta_2}{1 - \eta_2} \right] \cdot \Re \left[\frac{1 + \eta_3}{1 - \eta_3} \right] \\ &\quad - \sum_{\alpha_1=1, \alpha_2=1, \alpha_3=0}^{\infty} \left(1 - \frac{\alpha!}{|\alpha|!}\right) (\eta_1^{\alpha_1} - \bar{\eta}_1^{\alpha_1})(\eta_2^{\alpha_2} - \bar{\eta}_2^{\alpha_2})(\eta_3^{\alpha_3} + \bar{\eta}_3^{\alpha_3}). \end{aligned}$$

(Note that there is a plus in the last factor to keep L' real.) Again, a computer search can find r so that L' is positive on $r\mathbb{D}^3$, and $r = .39$ works, so $R_3 < 2.6$. \square

Combining Propositions 4.2, 4.4 and 4.5, we get the main result of this section.

Theorem 4.6. *There are positive constants M_n and R_n such that whenever $T = (T_1, T_2, \dots, T_n) \in \mathcal{B}(H)^n$ satisfies (4.1) and $p(z)$ is a polynomial in n variables, then*

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{R_n \mathbb{D}^n} \quad (4.11)$$

$$\|p_{\text{sym}}(T)\| \leq M_n \|p\|_{\mathbb{D}^n}. \quad (4.12)$$

Moreover, one can choose $R_2 = 1.85$, $R_3 = 2.6$, $M_2 = 4.1$ and $M_3 = 16.6$.

Remark 4.7. Another way to estimate $\|p_{\text{sym}}(T)\|$, under the assumption (4.2), would be to crash through with absolute values. Let $\Delta_n = \{z \in \mathbb{C}^n : \sum_{j=1}^n |z_j| \leq 1\}$ and let r_n denote the Bohr radius of Δ_n , i.e. the largest r such that whenever $p(z) = \sum c_\alpha z^\alpha$ has modulus one on Δ_n , then $q(z) = \sum |c_\alpha| z^\alpha$ has modulus bounded by one on $r\Delta_n$. One then has the estimate that, under the hypothesis (4.2), and writing $C_n = 1/r_n$,

$$\|p_{\text{sym}}(T)\| \leq \|q\|_{\Delta_n} \leq \|p\|_{C_n \Delta_n}. \quad (4.13)$$

It was shown by L. Aizenberg [1, Thm. 9] that

$$\frac{1}{3e^{1/3}} < r_n \leq \frac{1}{3}.$$

So the estimate in (4.11) for pairs satisfying (4.2) does not follow from (4.13).

5 n -tuples of contractions

In an attempt to use the above technique for tuples $T \in \mathcal{B}(H)^n$ such that $\max_{1 \leq j \leq n} \|T_j\| \leq 1$, we consider restricting ζ to belong to Δ_n , and we replace σ by some probability measure μ supported on Δ_n .

Suppose we can find some function q such that

$$\Lambda_\mu(q)(z) := \int_{\Delta_n} q(\zeta) \Re \frac{1 + \bar{\zeta} \cdot z}{1 - \bar{\zeta} \cdot z} d\mu(\zeta) \quad (5.1)$$

equals $p(z)$. We do not actually need q to be a polynomial; having an absolutely convergent power series on Δ_n (in ζ and $\bar{\zeta}$) is enough.

Lemma 5.1. *With notation as above, assume $\Lambda_\mu(q) = p$ and that $T \in \mathcal{B}(H)^n$ is an n -tuple of contractions. Then*

$$\|(p)_{\text{sym}}(T)\| \leq \|q\|_{\text{suppt}(\mu)} \leq \sup\{|q(z)| : z \in \Delta_n\}.$$

Proof. We assume first that $\max_{1 \leq j \leq n} \|T_j\| < 1$ and use the notation $K(\zeta, T)$ from the proof of Proposition 4.2 (which is permissible as $\|\zeta \cdot T\| < 1$ for $\zeta \in \Delta_n$). We have

$$(\Lambda_\mu q)_{\text{sym}}(T) = \int_{\Delta_n} q(\zeta) K(\bar{\zeta}, T) d\sigma(\zeta)$$

and hence the inequality $\|(p)_{\text{sym}}(T)\| \leq \|q\|_{\text{suppt}(\mu)}$ follows as in the previous proof.

If $\max_{1 \leq j \leq n} \|T_j\| = 1$, we deduce the result from $\|(p)_{\text{sym}}(rT)\| \leq \|q\|_{\Delta_n}$ for $0 < r < 1$. \square

Remark 5.2. For an arbitrary measure μ , there might be no q such that $\Lambda_\mu(q) = p$. If μ is chosen to be circularly symmetric, though, one gets

$$\Lambda_\mu(z^\alpha) = \left[\frac{|\alpha|!}{\alpha_1! \dots \alpha_n!} \int |\zeta^\alpha|^2 d\mu(\zeta) \right] z^\alpha. \quad (5.2)$$

As long as none of the moments on the right of (5.2) vanish, inverting Λ_μ is now straightforward.

To make use of the lemma to bound $p_{\text{sym}}(T)$ we need to find a way to choose another polynomial q and a μ on Δ_n so that $p = \Lambda_\mu q$ and $\|q\|_{\Delta_n}$ is small. We do not know a good way to do this.

Question 1. What is the smallest constant R_n such that, for every n -tuple T of contractions and every polynomial p , one has

$$\|p_{\text{sym}}(T)\| \leq \|p\|_{R_n \mathbb{D}^n} \quad (5.3)$$

We do not know if one can choose R_n smaller than the reciprocal of the Bohr radius of the polydisk, even when $n = 2$.

Question 2. Is there a constant M_n such that, for every n -tuple T of contractions and every polynomial p , one has

$$\|p_{\text{sym}}(T)\| \leq M_n \|p\|_{\mathbb{D}^n} ? \quad (5.4)$$

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